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On the quantum spectrum of isochronous potentials

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Abstract

In this paper, the quantum spectrum of isochronous potentials is investigated. Given that the frequency of the classical motion in such potentials is energy independent, it is natural to expect their quantum spectra to be equispaced. However, as has already been shown in some specific examples, this property is not always true. To gain some general insight into this problem, a WKB analysis of the spectrum, valid for any analytic potential, is performed and the first semiclassical corrections to its regular spacing are calculated. We illustrate the results on the two-parameter family of isochronous potentials derived in Stillinger and Stillinger (1989 *J. Phys. Chem.* **93** 6890), which includes the harmonic oscillator, the asymmetric parabolic well, the radial-harmonic oscillator and Urabe's potential as special limiting cases. In addition, some new analytical expressions for families of isochronous potentials and their corresponding spectra are derived by means of the above-mentioned method.

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1. Introduction

Several studies have already been devoted to the interesting problem of the relation between classical and quantum 'generalized harmonicity'. In classical mechanics, the motion of an oscillator in the parabolic well $V(x) = \omega^2 x^2/2$ is known to possess an energy-independent frequency ω. However, this harmonicity is not specific to the latter and can be achieved by designing adequate potentials called *isochronous*. Their construction simply amounts to shearing the parabolic well in such a way that, at fixed energy, the distance between two turning points is preserved (see for example [2, 3]).

In quantum mechanics, the energy levels of a parabolic well are regularly spaced by a quantity $\hbar\omega$. Thus, a generalized harmonicity is naturally defined by an equispaced spectrum. Similar to the classical case, it is possible to construct potentials, essentially different from the parabolic well, whose spectrum is exactly harmonic. This can be done by applying a supersymmetric transform followed by a Darboux transform to the harmonic well which lead to families of potentials called *isospectral* partners of the harmonic oscillator (for a review, see for example [4]). A famous example of such a potential has been derived by Abraham and Moses [5].

A link between these classical and quantum transformations has been established by Eleonskii *et al* [6]. They show that the classical limit of the isospectral transformation is precisely the isochronism preserving shear mentioned previously. The next question is whether the two classes of classical and quantum generalized harmonic potentials are the same. Surprisingly enough, they are not. Both aspects of this problem have been investigated by several authors who demonstrated through specific examples that neither isochronism generates an equispaced spectrum nor does a regularly spaced spectrum stem from an isochronous potential in general [1, 7–10].

Despite this inequivalence, it is worth noting that the semiclassical Einstein–Brillouin–Keller (EBK) quantization of an isochronous potential leads to an equispaced EBK spectrum. Indeed, the semiclassical quantization rule is $I_0(E) = \hbar\left(n+\frac{1}{2}\right)$, $n \in \mathbb{N}$, where the action $I_0(E)$ is related to the frequency $\omega(E)$ and the energy E by $\mathrm{d}I_0/\mathrm{d}E = \omega(E)^{-1}$. As the frequency of an isochronous potential is energy independent, $E = \omega I_0$ and EBK levels are given by $E_n^{\mathrm{EBK}} = \left(n+\frac{1}{2}\right)\hbar\omega$. Thus, their spacing is constant, $E_{n+1}^{\mathrm{EBK}} - E_n^{\mathrm{EBK}} = \hbar\omega$.

Consequently, as the usual semiclassical quantization fails to distinguish between the spectrum of an isochronous potential and a purely harmonic spectrum, studies to date have resorted either to the study of isochronous potentials whose quantum spectrum (or quantization condition) is exactly known or to numerics. To date, the asymmetric parabolic (also called *split-harmonic*) well[1] is the only isochronous potential whose exact quantization condition proves its quantum levels to be non-strictly equidistant [1, 8]. Another class of potentials defined on a finite interval of the real axis, and first derived by Urabe [11], has been investigated analytically by Mohazzabi [10]. But no closed form for the quantization is available in this case and the lack of regularity in the spacings has been computed numerically.

Finally, Stillinger and Stillinger [1] have proposed a two-parameter family of isochronous potentials interpolating between the harmonic oscillator and the asymmetric parabolic well. Although not explicitly mentioned in their paper, this family includes the radial-harmonic potential (also known as the *isotonic* potential) and Urabe's potential. For non-extremal values of the parameters, this family is analytic on the entire real line. It also includes the one-parameter family of analytic potentials defined by Bolotin and MacKay [3]. To prove that the exact spectrum of these potentials is not equidistant in general, the authors based their argument on the assumption that energy levels depend continuously on the parameters. As the family interpolates continuously between the harmonic oscillator, whose spectrum is equispaced, and the asymmetric parabolic well, whose spectrum is not equispaced, they deduced that the spectrum is not exactly equidistant for some range of the parameters. However, this result is only qualitative in the sense that the authors do not provide a method to evaluate the spectrum for general values of the parameters.

The aim of the present paper is precisely to fill in this gap and to provide a way to calculate the first corrections to the equispacing of the EBK spectrum. In section 2, we first review some properties of isochronous potentials and derive *new analytical expressions* for some one- or two-parameter families of isochronous potentials. We then use the perturbation method WKB up to fourth order (beyond the EBK quantization) to derive an expression for the corrections to the equispacing valid for any analytic isochronous potential (section 3). In section 4, we apply these results to the potentials described above and derive an analytical or asymptotic expression for their *higher order WKB* quantization condition. The resulting higher order WKB spectra are then checked against numerical evaluations of the exact quantum problem

[1] See [1], the latter has the form $V(x) = \omega_1^2 x^2/2$ for $x < 0$ and $V(x) = \omega_2^2 x^2/2$ for $x > 0$.

and prove not only to reproduce the right behaviour of the quantum levels at high energy but also to be quite accurate at low energy for some range of the parameters. A discussion of these results and some conclusions are given in section 6.

2. Isochronous potentials

2.1. Generalities

The literature about isochronous potentials (or more generally about isochronous centres) is particularly vast and the interested reader is referred to [13] and references therein for more information. This section is not intended to present a detailed analysis of isochronism but merely to recall some basic facts and to introduce certain notations which will be useful in the rest of this paper. Our approach of isochronism is mainly inspired by [14] and based on the so-called *S function* (see below for a definition). Another fruitful way to obtain explicit analytical expressions for isochronous potentials has been developed in [15]. Finally, the reader interested in *multidimensional* or *many-body* isochronous systems should consult [16] and references therein.

A potential $V(x)$ is said to be isochronous if it generates a motion $x(t)$, obeying $\ddot{x} + V'(x) = 0$, whose period (frequency) is energy independent. It can be shown that such a potential has a single minimum (see for instance [3]). For potentials which are at least twice differentiable at the origin, we will consider $V(0) = 0$, $V'(0) = 0$ and $V''(0) = \omega^2$ throughout, without loss of generality.

For arbitrary potentials, the period of motion is

$$T(E) = \sqrt{2}\int_{x_-(E)}^{x_+(E)} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}, \tag{1}$$

where $x_\pm(E)$ are the two turning points of the trajectory at energy E. Conversely, it has been shown by Landau and Lifshitz [12] that, for a prescribed period $T(E)$, there is a corresponding distance between the turning points given by

$$x_+(E) - x_-(E) = \frac{1}{\pi\sqrt{2}}\int_0^E \frac{T(u)}{\sqrt{E-u}}\,\mathrm{d}u. \tag{2}$$

Imposing isochronism to the potential ($T(E) = \mathrm{cst} = 2\pi/\omega$) yields

$$x_+(E) - x_-(E) = 2\frac{\sqrt{2E}}{\omega}. \tag{3}$$

If we let $\hat{x}(E)$ be the middle of the segment $[x_-(E), x_+(E)]$, we obtain the equivalent formulation

$$x_\pm(E) = \pm\frac{\sqrt{2E}}{\omega} + \hat{x}(E). \tag{4}$$

An entire freedom is left in the choice of $\hat{x}(E)$ provided the inversion of (4) leads to a potential $V(x)(\equiv E)$ single valued on the real axis. The smoothness of $\hat{x}$ induces the smoothness of V. Geometrically, the function $\hat{x}(E)$ defines a *shear* of the parabolic potential giving rise to another isochronous potential (for an illustration of such a transformation see [2]).

For our purpose, it is convenient to rephrase the results above in a slightly different way. Let us define a variable $X \in \mathbb{R}$ through the transformation

$$V(x) = \frac{1}{2}\omega^2 X^2, \qquad \frac{\mathrm{d}x}{\mathrm{d}X} > 0. \tag{5}$$

The two equations (4) can now be recast into a single one

$$x(X) = X + \bar{x}(X). \tag{6}$$

The function $x(X)$ now represents the two branches of the potential V parametrized by X. The new shear function $\bar{x}$ is related to $\hat{x}$ by $\bar{x}(X) = \hat{x}\left(\frac{1}{2}\omega^2 X^2\right)$ and is even in X,

$$\bar{x}(X) = \bar{x}(-X). \tag{7}$$

As $V(0) = 0$, equations (5) and (6) yield $x(0) = \bar{x}(0) = 0$. The condition $\mathrm{d}x/\mathrm{d}X > 0$ of (5) ensures that X and x are in bijection, i.e. $V(x)$ is single valued. Note that we could have chosen $\mathrm{d}x/\mathrm{d}X < 0$ instead. Now, from (6),

$$\frac{\mathrm{d}x}{\mathrm{d}X} = 1 + S(X), \tag{8}$$

where

$$S(X) = \frac{\mathrm{d}\bar{x}}{\mathrm{d}X}. \tag{9}$$

The latter is similar to the function introduced in [14] but our definition of X does not include the frequency ω.

Then, $\mathrm{d}x/\mathrm{d}X > 0 \Rightarrow S(X) > -1$. Moreover, as $\bar{x}(X)$ is even, $S(X)$ is odd,

$$S(X) = -S(-X), \tag{10}$$

which finally yields

$$\forall X \in \mathbb{R}, \qquad |S(X)| < 1. \tag{11}$$

Note that the function S defines the potential V up to a multiplicative constant only. Indeed, from (8) we have

$$x = X + \int_0^X S(u)\,\mathrm{d}u. \tag{12}$$

Inverting this relation gives $X(x)$ and thus, from (5),

$$V(x) = \tfrac{1}{2}\omega^2[X(x)]^2. \tag{13}$$

The multiplicative constant is thus basically the square of the frequency, which can be chosen arbitrarily.

2.2. Scaling properties

In what follows, we derive some simple, though important, results regarding the scaling properties of isochronous potentials.

Claim 1. *Let $V(x)$ be isochronous with frequency ω. Then,*

$$\forall(\gamma,\beta) \in \mathbb{R}^{*2}, \qquad \tilde{V}(x) = \left(\frac{\gamma}{\beta}\right)^2 V(\beta x) \tag{14}$$

is isochronous with frequency $\tilde{\omega} = \gamma\omega$.

Proof. Let $I_0(E)$ be the action associated with $V(x)$. Then, $I_0(E) = \frac{\sqrt{2}}{\pi}\int_{x_-(E)}^{x_+(E)} \times \sqrt{E - V(x)}\,\mathrm{d}x$, where $V(x_\pm(E)) = E$. Let $\tilde{I}_0(E)$ be associated with $\tilde{V}(x)$. Its turning points at energy E are related to those of $V(x)$ by

$$\tilde{x}_\pm(E) = \frac{1}{\beta} x_\pm\left(\frac{\beta^2 E}{\gamma^2}\right). \tag{15}$$

Hence,

$$\tilde{I}_0(E) = \frac{\gamma}{\beta^2} I_0\left(\frac{\beta^2 E}{\gamma^2}\right). \tag{16}$$

So far, this relation is true for all potentials.

For an isochronous potential, however, $I_0(E) = E/\omega$. Then, from the previous relation, $\tilde{I}_0(E) = E/(\gamma\omega)$, which proves that $\tilde{V}(x)$ is isochronous with frequency $\gamma\omega$. □

Corollary. *If $V(x)$ is isochronous with frequency ω,*

$$\tilde{V}(x) = \frac{1}{\beta^2} V(\beta x) \tag{17}$$

is isochronous with the same frequency.

This shows that, once an isochronous potential $V(x)$ is known, a one-parameter family of isochronous potentials with the same frequency may be derived from the above scaling.

Claim 2. *The functions $\tilde{\bar{x}}(X)$ and $\tilde{S}(X)$ corresponding to $\tilde{V}(x)$ are related to $\bar{x}(X)$ and $S(X)$ defining $V(x)$ by*

$$\tilde{\bar{x}}(X) = \frac{1}{\beta}\bar{x}(\beta X) \qquad \text{and} \qquad \tilde{S}(X) = S(\beta X). \tag{18}$$

Proof. By definition $\tilde{x}(X) = X + \tilde{\bar{x}}(X)$, which yields $\tilde{\bar{x}}(X) = [\tilde{x}(X) + \tilde{x}(-X)]/2$. Starting from $\tilde{V}(\tilde{x}) = \frac{1}{2}\tilde{\omega}^2 X^2$, we obtain $(\gamma/\beta)^2 V(\beta\tilde{x}) = \frac{1}{2}\gamma^2\omega^2 X^2$, that is $V(\beta\tilde{x}) = \frac{1}{2}\omega^2(\beta X)^2$. Hence, $\tilde{x}(X) = \frac{1}{\beta}x(\beta X)$. Reinstating this last expression in $\tilde{\bar{x}}(X)$ and using $\bar{x}(X) = [x(X) + x(-X)]/2$, we get the first of the relations (18). The second is obvious due to the relation $\tilde{S}(X) = \mathrm{d}\tilde{\bar{x}}(X)/\mathrm{d}X$. □

Note that, as expected, β is the parameter involved in the scaling relations (18) while γ is an overall prefactor responsible for the tuning of the frequency only.

Claim 3. *Asymptotic behaviour of $\tilde{V}(x)$ defined by (17) with respect to β. Provided $S(X)$ is defined on the entire real line,*

$$\text{as } \beta \to 0, \qquad \tilde{V}(x) \to \frac{1}{2}\left(\frac{\omega}{1 + S_0^+ \,\mathrm{sgn}\,(x)}\right)^2 x^2, \tag{19}$$

$$\text{as } \beta \to \infty, \qquad \tilde{V}(x) \to \frac{1}{2}\left(\frac{\omega}{1 + \langle S\rangle \,\mathrm{sgn}\,(x)}\right)^2 x^2, \tag{20}$$

where

$$S_0^+ = \lim_{X\to 0^+} S(X) \qquad \text{and} \qquad \langle S\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T S(u)\,\mathrm{d}u.$$

The function $\mathrm{sgn}(x)$ is equal to 1 if $x > 0$ and -1 if $x < 0$. If $S(X)$ is continuous in $X = 0$, then $S(0) = 0$. The first expression then reduces to the harmonic oscillator potential. The second one corresponds to a split-harmonic oscillator whose frequencies are $\omega_\pm = \omega/(1 \pm \langle S\rangle)$.

Proof. Let us rewrite equation (12) in the form

$$x = X\left(1 + \frac{1}{X}\int_0^X S(u)\,\mathrm{d}u\right).$$

Now, applied to the rescaled potential for which $\tilde{S}(X) = S(\beta X)$ (see (18)), this gives

$$x = X\left(1 + \frac{1}{X}\int_0^X \tilde{S}(u)\,\mathrm{d}u\right) = X\left(1 + \frac{1}{\beta X}\int_0^{\beta X} S(v)\,\mathrm{d}v\right).$$

Taking the limits $\beta \to 0^+$ and $\beta \to \infty$ yield $x = X\big(1 + S_0^+ \operatorname{sgn}(X)\big)$ and $x = X(1 + \langle S\rangle \operatorname{sgn}(X))$, respectively, and, as $\operatorname{sgn}(X) = \operatorname{sgn}(x)$, we obtain (19) and (20). □

Now, using the results of the previous proof, we can obtain the asymptotic behaviour of an isochronous potential defined on the entire real line when $|x| \to \infty$.

Claim 4. *Providing $|\langle S\rangle| \neq 1$, where $\langle S\rangle$ is defined in claim 3,*

$$V(x) = \frac{1}{2}\left(\frac{\omega}{1 + \langle S\rangle \operatorname{sgn}(x)}\right)^2 x^2 + o(x^2), \qquad |x| \to \infty. \tag{21}$$

When $|\langle S\rangle| = 1$, the potential is singular. In this case, only one branch (say $x \to \infty$) is asymptotically parabolic such that $V(x) = \frac{\omega^2}{8}x^2 + o(x^2)$, $x \to \infty$.

2.3. Examples

We now apply the above considerations to the determination of isochronous potentials whose analytical expression can be given explicitly. We start from the function $S(X)$ which is required to satisfy the conditions given in (10) and (11). We then apply scaling (17) to derive a family of potentials with the same frequency. To our knowledge, the families of potentials derived in section 2.3.2 have never before appeared in the literature.

2.3.1. Family I. Let

$$S(X) = \frac{\alpha X}{\sqrt{1 + \alpha X^2}}, \qquad \alpha \in [0, 1]. \tag{22}$$

The condition on α ensures that $|S(X)| < 1$. Moreover, $S(X) = -S(-X)$, so that conditions (10) and (11) are fulfilled. The shear function corresponding to $S(X)$ is $\bar{x}(X) = \int_0^X S(u)\,\mathrm{d}u = \sqrt{1 + \alpha X^2} - 1$ and inverting (12) leads to

$$X = \frac{(x+1) - \sqrt{\alpha x(x+2) + 1}}{1 - \alpha}.$$

Using (13) we obtain the potential

$$V(\alpha; x) = \frac{\omega^2}{2}\left(\frac{(x+1) - \sqrt{\alpha x(x+2) + 1}}{1 - \alpha}\right)^2. \tag{23}$$

This is precisely the *one-parameter* family of potentials derived by Bolotin and MacKay in [3]. Note that the parameter α is distinct from the scaling parameter β introduced in the previous section. Thus, relation (17) enables us to derive a *two-parameter* family of isochronous potentials

$$V(\alpha, \beta; x) = \frac{\omega^2}{2\beta^2}\left(\frac{(\beta x+1) - \sqrt{\alpha\beta x(\beta x+2) + 1}}{1 - \alpha}\right)^2 \tag{24}$$

presented for the first time by Stillinger and Stillinger[2] in [1].

The family (24) includes several known isochronous potentials as limiting cases which are listed below.

[2] This is clear from the substitution $\omega^2 \to K$, $\alpha \to \xi^2$, $\beta \to \sqrt{\beta}/\xi$.

- $\alpha \to 0$ or $\beta \to 0$: *harmonic oscillator*.

$$V(x) = \frac{\omega^2}{2} x^2. \tag{25}$$

- $\alpha \to 1$ and $\beta \neq 0$: *isotonic potential*.

$$V(x) = \frac{\omega^2}{8\beta^2}\left(\beta x + 1 - \frac{1}{\beta x + 1}\right)^2, \qquad x > -\frac{1}{\beta}. \tag{26}$$

This potential, also called radial-harmonic oscillator, has the property to be singular at $x = -1/\beta$ and to be defined on a half-line.

- $\beta \to \infty$: *split-harmonic oscillator*.

$$V(x) = \begin{cases} \frac{1}{2}\left(\frac{\omega}{1+\sqrt{\alpha}}\right)^2 x^2, & x \geqslant 0, \\ \frac{1}{2}\left(\frac{\omega}{1-\sqrt{\alpha}}\right)^2 x^2, & x \leqslant 0. \end{cases} \tag{27}$$

In the case $\alpha \to 1$, the left part ($x \leqslant 0$) of the potential converges to a hard wall in $x = 0$ and gives rise to a *half-parabolic* potential.

- $\alpha\beta = \zeta \neq 0$ and $\beta \to 0$: *Urabe's potential*.

$$V(x) = \frac{\omega^2}{2\zeta^2}(1 - \sqrt{1 + 2\zeta x})^2, \qquad x \in \left[-\frac{1}{2\zeta}, \frac{3}{2\zeta}\right]. \tag{28}$$

Note that in this last case, the condition $\alpha \in [0, 1]$ has been relaxed. Preserving $|S(X)| < 1$ amounts to restricting the values of X to $|X| < 1/|\zeta|$. Hence, a potential defined only on a finite interval of the real axis.

Note that the first and third results can be obtained by means of the general relations (19) and (20).

2.3.2. Family II. Let

$$S(X) = \frac{1}{\xi}\frac{\sinh X}{\cosh X - 1 + \alpha}, \qquad (\alpha, \xi) \in \mathbb{R}^{*+}. \tag{29}$$

$S(X)$ is readily odd. According to the value of α, the condition $|S(X)| < 1$ imposes

$$\begin{cases} 0 < \alpha < 1 & \Rightarrow \quad \xi > [\alpha(2-\alpha)]^{-1/2}, \\ \alpha \geqslant 1 & \Rightarrow \quad \xi \geqslant 1. \end{cases} \tag{30}$$

The shear function of (29) is given by $\bar{x}(X) = \{\ln[(\cosh(X) - 1 + \alpha)/\alpha]\}/\xi$ and inverting (12) (or (6)) amounts to solving

$$Y^{\xi+1} + 2(\alpha - 1)Y^{\xi} + Y^{\xi-1} - 2\alpha\, e^{\xi x} = 0, \tag{31}$$

where $Y = \exp X$. For some particular values of α and ξ, the above algebraic equation can be solved exactly for Y. This leads to an analytical expression for the potential. In what follows, we work out some of the simplest cases and systematically apply the scaling (17) to derive the corresponding family.

- $\xi = 1$ and $\alpha \geqslant 1$.

$$V(x) = \frac{\omega^2}{2\beta^2}\left[\ln\left(1 - \alpha + \sqrt{2\alpha(e^{\beta x} - 1) + \alpha^2}\right)\right]^2,$$
$$x > -\frac{1}{\beta}\ln 2\alpha. \tag{32}$$

This represents a two-parameter family of potentials singular in $x = -(\ln 2\alpha)/\beta$ and defined on a half-line.

Note that for $\alpha = 1$, the analytical form of this potential simplifies further to yield the remarkably simple expression

$$V(x) = \frac{\omega^2}{8\beta^2}\ln^2(2\,e^{\beta x} - 1), \qquad x > -\frac{\ln 2}{\beta}. \tag{33}$$

- $\xi = 2$ and $\alpha = 1$.

$$V(x) = \frac{\omega^2}{2\beta^2}\left[\frac{2}{3}\beta x + \ln\left(q_+^{1/3} - q_-^{1/3}\right)\right]^2, \qquad q_\pm = \left(\sqrt{1 + \frac{e^{-4\beta x}}{27}} \pm 1\right). \tag{34}$$

Contrary to the previous family, this one is defined on the whole line $\mathbb{R}$ and is not singular.

- $\xi = 3$ and $\alpha = 1$.

$$V(x) = \frac{\omega^2}{8\beta^2}\left[\ln\left(\frac{\sqrt{1 + 8\,e^{3\beta x}} - 1}{2}\right)\right]^2 \tag{35}$$

is defined on the whole line $\mathbb{R}$ and is not singular, as in the previous case.

Due to Cardano's formula, (31) can be solved analytically in the more general case, where ξ is either equal to 2 or 3 and $\alpha > 1$. For the same reason, it is also possible to derive exact solutions of (31) when $\alpha = 1$ and $\xi = 5$ or 7. These expressions, however, are somewhat messy and are not provided. It is also possible to derive exact solutions of (31) when $\alpha = 2$ and $\xi = 5$ or 7. The potentials given above are a few among many others.

Whatever the values of the parameters ξ and α, the limiting case $\beta \to 0$ leads to the harmonic potential as $S(X)$ given by (29) is continuous in $X = 0$. Given that $\langle S\rangle = 1/\xi$, the limiting case $\beta \to \infty$ gives rise to a split-harmonic oscillator whose left and right frequencies are $\omega_\pm = \xi\omega/(\xi \pm 1)$. In the first case, where $\xi = 1$, this leads to a half-parabolic well.

3. Beyond EBK

We now turn to the problem of the determination of the quantum spectrum of isochronous potentials. As already stated in the introduction, their semiclassical EBK spectrum (i.e., obtained from a first-order WKB method) is perfectly regularly spaced. Nevertheless, a study of the exact spectrum of the split-harmonic oscillator leads to the conclusion that at least not all isochronous potentials possess strictly equidistant energy levels [1, 8]. As the determination of an exact quantization condition is not possible in general, a natural idea is to go beyond the usual semiclassical approximation and to study the properties of the spectrum by means of the higher order terms generated by the WKB method.

The WKB method to all orders has been first developed by Dunham [17] and subsequently improved by many authors [18–20, 22]. In particular, it allows the quantization condition for a 1D analytic potential to be written as a power series in $\hbar$. Such series have to be interpreted as asymptotic series and are generally not convergent. However, the result leads to the exact quantization condition for those exactly solvable potentials, whose WKB series can be evaluated explicitly and summed.

In what follows, we briefly recall the WKB method before making use of the second and fourth terms derived in [19] to express the first corrections to the EBK quantization in terms of the S function presented in the previous section.

3.1. WKB to all orders

For a complete description of the method and the properties of WKB series, the reader is referred to [18–20, 22]. The present introduction is mainly inspired by [19]. We start from the Schrödinger equation in 1D ($m = 1$),

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi(x) = E\psi(x). \tag{36}$$

Writing

$$\psi(x) = \exp\left(\frac{\mathrm{i}}{\hbar}\sigma(x)\right), \tag{37}$$

we obtain

$$\sigma'^2(x) + \left(\frac{\hbar}{\mathrm{i}}\right)\sigma''(x) = 2(E - V(x)), \tag{38}$$

solved by means of a power series in $\hbar$,

$$\sigma(x) = \sum_{k=0}^{\infty}\left(\frac{\hbar}{\mathrm{i}}\right)^k \sigma_k(x). \tag{39}$$

Reinstating (39) in (38) and solving order by order in $\hbar$ yields the recurrence

$$\sigma_0'^2 = 2(E - V) \qquad \text{and} \qquad \sum_{k=0}^{l}\sigma_k'\sigma_{l-k}' + \sigma_{l-1}' = 0, \qquad l \geqslant 1. \tag{40}$$

Requiring the wavefunction to be single valued amounts to imposing the following quantization condition:

$$\oint_\gamma \mathrm{d}\sigma = \sum_{k=0}^{\infty}\left(\frac{\hbar}{\mathrm{i}}\right)^k \oint_\gamma \mathrm{d}\sigma_k = 2\pi\hbar n, \qquad n \in \mathbb{N}, \tag{41}$$

where the complex integration contour γ surrounds the two turning points of $V(x)$ at energy E located on the real axis.

The first term of the series (41) is readily proportional to the classical action $I_0(E)$

$$\oint_\gamma \mathrm{d}\sigma_0 = 2\int_{x_-(E)}^{x_+(E)} \sqrt{2(E - V(x))}\,\mathrm{d}x = 2\pi I_0(E). \tag{42}$$

As integral of a logarithmic derivative, the second term can be shown to be equal to

$$\left(\frac{\hbar}{\mathrm{i}}\right)\oint_\gamma \mathrm{d}\sigma_1 = -\pi\hbar. \tag{43}$$

If we truncate the WKB series to this last order, we re-derive the so-called EBK quantization

$$I_0(E) = \left(n + \tfrac{1}{2}\right)\hbar, \qquad n \in \mathbb{N}, \tag{44}$$

where the $1/2$ term represents the Maslov index.

As shown by Fröman and Fröman [24], all odd terms σ_{2k+1}', $k \geqslant 1$, are total derivatives and as such, their contribution vanishes. This allows us to rewrite the quantization condition (41) as

$$\sum_{k=0}^{\infty} I_{2k}(E) = \left(n + \tfrac{1}{2}\right)\hbar, \qquad n \in \mathbb{N}, \tag{45}$$

where we have defined

$$I_{2k}(E) = \frac{1}{2\pi}\left(\frac{\hbar}{\mathrm{i}}\right)^{2k}\oint_\gamma \mathrm{d}\sigma_{2k}, \qquad k \in \mathbb{N}. \tag{46}$$

In case $V(x)$ is analytic and $V'(x) \neq 0$ if $x \neq 0$, the authors of [22] have proved that the contour integrals involved in (46) can be systematically replaced by equivalent Riemann integrals between the two turning points. Thanks to this result, it can be shown that [19]

$$I_2(E) = -\frac{\hbar^2}{24\sqrt{2}\pi}\frac{\partial^2}{\partial E^2}\int_{x_-(E)}^{x_+(E)} \mathrm{d}x\,\frac{V'^2(x)}{\sqrt{E-V(x)}} \tag{47}$$

and

$$I_4(E) = \frac{\hbar^4}{4\sqrt{2}\pi}\left[\frac{1}{120}\frac{\partial^3}{\partial E^3}\int_{x_-(E)}^{x_+(E)} \mathrm{d}x\,\frac{V''^2(x)}{\sqrt{E-V(x)}} - \frac{1}{288}\frac{\partial^4}{\partial E^4}\int_{x_-(E)}^{x_+(E)} \mathrm{d}x\,\frac{V'^2(x)V''(x)}{\sqrt{E-V(x)}}\right]. \tag{48}$$

As the I_{2k} are originally given by contour integrals, addition of total derivatives to the σ'_{2k} does not change their value. However, it modifies their expression in terms of the potential and its derivatives and possibly allows for their simplification. Such a technique is developed and used systematically in [22]. Expressions (47) and (48) are two among others (see for instance [21]).

3.2. WKB expansion for isochronous potentials

We now use the results obtained in section 2.1 to express I_2 and I_4 in terms of the function $S(X)$ related to the potential $V(x)$. We make use of the change of variables $V(x) = \omega^2 X^2/2$ and of the relation (8) to find

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\omega^2 X}{1+S(X)}. \tag{49}$$

Noting further that $X(x_\pm(E)) = \sqrt{2E}/\omega$, we obtain

$$I_2(E) = -\frac{\hbar^2\omega^4}{24\sqrt{2}\pi}\frac{\partial^2}{\partial E^2}\int_{-\frac{\sqrt{2E}}{\omega}}^{\frac{\sqrt{2E}}{\omega}} \frac{X^2\,\mathrm{d}X}{(1+S(X))\sqrt{E-\frac{1}{2}\omega^2X^2}}. \tag{50}$$

Then, using the fact that $S(X)$ is odd and making the change of variables $u = \omega^2X^2/2$, we find

$$I_2(E) = -\frac{\hbar^2\omega}{12\pi}\frac{\partial^2}{\partial E^2}\left[E\int_0^1 \frac{u^{1/2}}{(1-u)^{1/2}}\frac{\mathrm{d}u}{1-S^2\left(\frac{\sqrt{2Eu}}{\omega}\right)}\right]. \tag{51}$$

Another equivalent formulation suitable for an asymptotic analysis is given by

$$I_2(E) = -\frac{\hbar^2\omega}{12\pi}\frac{1}{E^2}\int_0^E \frac{v^{3/2}}{(E-v)^{1/2}}\frac{\mathrm{d}^2}{\mathrm{d}v^2}\left[\frac{v}{1-S^2\left(\frac{\sqrt{2v}}{\omega}\right)}\right]\mathrm{d}v. \tag{52}$$

This way, I_2 is expressed through an Abel type (or Riemann–Liouville fractional) integral about which a lot is known [27, 32].

A similar calculation yields the fourth-order correction

$$I_4(E) = \frac{\hbar^4}{4\pi\omega}\left[\frac{1}{120}\frac{\partial^3}{\partial E^3}\int_0^1 \frac{G_1\left(\frac{\sqrt{2Eu}}{\omega}\right)}{u^{1/2}(1-u)^{1/2}}\,\mathrm{d}u - \frac{1}{288}\frac{\partial^4}{\partial E^4}\int_0^1 \frac{G_2\left(\frac{\sqrt{2Eu}}{\omega}\right)}{u^{1/2}(1-u)^{1/2}}\,\mathrm{d}u\right], \tag{53}$$

where the functions G_1 and G_2 are given by

$$G_1(X) = \frac{\omega^4}{(1-S^2)^3}\left[3S^2+1+\frac{8XSS'(1+S^2)}{1-S^2}+\frac{X^2S'^2(1+10S^2+5S^4)}{(1-S^2)^2}\right],$$
$$G_2(X) = \frac{\omega^6X^2}{(1-S^2)^3}\left[3S^2+1+\frac{4XSS'(1+S^2)}{1-S^2}\right]. \tag{54}$$

In the last expressions, S stands for $S(X)$ and S' for $\mathrm{d}S/\mathrm{d}X$.

Similar to I_2, I_4 is equivalently expressed through Abel integrals via

$$I_4(E) = \frac{\hbar^4}{4\pi\omega}\left[\frac{E^{-3}}{120}\int_0^E \frac{v^{5/2}}{(E-v)^{1/2}}\frac{\mathrm{d}^3}{\mathrm{d}v^3}\left\{G_1\left(\frac{\sqrt{2v}}{\omega}\right)\right\}\mathrm{d}v - \frac{E^{-4}}{288}\int_0^E \frac{v^{7/2}}{(E-v)^{1/2}}\frac{\mathrm{d}^4}{\mathrm{d}v^4}\left\{G_2\left(\frac{\sqrt{2v}}{\omega}\right)\right\}\mathrm{d}v\right]. \tag{55}$$

3.3. Scaling properties of WKB series

We have seen in section 2.2 that the general scaling $\tilde{V}(x) = (\gamma/\beta)^2V(\beta x), (\gamma,\beta) \in \mathbb{R}^{*2}$, preserves isochronism and changes the frequency ω of $V(x)$ to $\tilde{\omega} = \gamma\omega$. Such a transformation, however, is not expected to have simple consequences at the quantum level given the lack of relation between the spectra of $V(x)$ and $\tilde{V}(x)$ in the general case. This can be seen directly from the Schrödinger equation itself

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\tilde{V}(x)\right]\psi(x) = \tilde{E}\psi(x), \tag{56}$$

$$\Leftrightarrow \quad \left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\gamma^2}{\beta^4}V(y)\right]\varphi(y) = \frac{\tilde{E}}{\beta^2}\varphi(y), \tag{57}$$

where $y = \beta x$ and $\varphi(y) = \psi(x)$.

This last equation is equivalent to the Schrödinger equation for V iff $\gamma = \pm\beta^2$, which corresponds to the particular scaling $\tilde{V}(x) = \beta^2V(\beta x)$. The spectrum of $\tilde{V}$ is then the spectrum of V multiplied by β^2.

But nothing can be said about the general transformation and the frequency-preserving scaling $\tilde{V}(x) = V(\beta x)/\beta^2$ in particular without further information regarding the potential $V(x)$.

As we are dealing with isochronous potentials, we may transform the Schrödinger equation (36) into an equivalent Sturm–Liouville problem involving the function $S(X)$ rather than the potential $V(x)$ itself. Thanks to (5) and (8), we obtain

$$\frac{\hbar^2}{2}\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{1}{1+S}\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)+(1+S)\left[E-\frac{\omega^2}{2}X^2\right]\phi = 0, \tag{58}$$

where $\phi \equiv \phi(X) = \psi(x)$ and $S \equiv S(X)$. This last form proves convenient for numerical purposes as $S(X)$ is a supplied function whose analytical form can be chosen freely within the requirements (10) and (11). But we found it of no particular help in investigating scaling effects on the spectrum.

Instead, we can look at the way this scaling affects each term of the WKB series.

Claim 5. *Let I_{2n} and $\tilde{I}_{2n}$ be the $(2n)$th terms of the WKB series for the potentials $V(x)$ and $\tilde{V}(x) = (\gamma/\beta)^2 V(\beta x)$, respectively. Then,*

$$\tilde{I}_{2n}(E) = \left(\frac{\beta^2}{\gamma}\right)^{2n-1} I_{2n}\left(\frac{\beta^2 E}{\gamma^2}\right), \qquad n \in \mathbb{N}. \tag{59}$$

Note that this expression is *valid for any potential* and not only for isochronous potentials. Details of the proof are given in appendix A. But let us immediately note that multiplying equation (57) by β^4/γ^2 indicates that the energy is rescaled by β^2/γ^2 and $\hbar$, by β^2/γ. Since each term of the WKB series is proportional to a given power of $\hbar$, the above result is easily established.

Thus, we arrive to the conclusion that, even though no simple transform enables us to derive the spectrum of $\tilde{V}$ from the spectrum of V, each term of the WKB series obeys its own scaling, which allows for calculating the WKB series for $\tilde{V}$ once it is known for V. We may restrict our investigation to potentials with a fixed frequency and a fixed value for β.

4. Application

We now apply what precedes to some isochronous potentials presented in section 2. We start our investigation with the family of potentials derived by Bolotin and MacKay [3] and obtain an analytical expression for its fourth-order WKB quantization condition. This result is immediately generalized to the two-parameter Stillinger's family (family I), thanks to the scaling (59).

Regarding the class of potentials presented in 2.3.2 (family II), we would not be able to derive any explicit analytical expression for the corrections. We will nevertheless derive their asymptotic behaviour.

4.1. Family I

For the sake of simplicity, we calculate the terms I_2 and I_4 for a potential of frequency $\omega = \sqrt{2}$ whose S function is given by

$$S(X) = \sqrt{1-\eta}\frac{X}{\sqrt{1+X^2}}, \qquad \eta \in [0, 1]. \tag{60}$$

We recover a general expression of the type (24) for the potential by means of (18) with $\tilde{S}(X) = S(\beta\sqrt{\alpha}X)$ and $\eta = 1 - \alpha$. The frequency is then fixed to the value ω, thanks to $\gamma = \omega/\sqrt{2}$.

4.1.1. Analytic expressions for I_2 and I_4. Setting $\hbar = 1$, expression (51) leads to

$$I_2(E) = -\frac{\sqrt{2}}{2^4}\frac{(1-\eta)}{(1+\eta E)^{5/2}} \tag{61}$$

and expression (53) leads to

$$I_4(E) = \frac{\sqrt{2}}{2^{10}}\frac{(1-\eta)p(\eta, E)}{(1+\eta E)^{15/2}}, \tag{62}$$

where

$$p(\eta, E) = 15\eta^4 E^4 - 30\eta^3(7\eta - 16)E^3 - 3\eta^2(119\eta^2 - 651\eta + 411)E^2 \\ + 3\eta(455\eta^2 - 567\eta + 104)E - 280\eta^2 + 140\eta + 8.$$

4.1.2. WKB quantization condition to fourth order. The fourth-order WKB quantization condition (45) for the family of potentials (24) is deduced from (61) and (62) and from the scaling relation (59)

$$E - \frac{1}{8}\frac{(\alpha\beta)^2}{Q^{5/2}} + \frac{\alpha^4\beta^6}{2^8\omega^2}\frac{P}{Q^{15/2}} = \left(n + \frac{1}{2}\right)\omega, \tag{63}$$

where

$$Q = \left[1 + 2\alpha(1-\alpha)\frac{\beta^2}{\omega^2}E\right], \qquad P = p\left(1-\alpha, 2\alpha\frac{\beta^2}{\omega^2}E\right). \tag{64}$$

We can now solve equation (63) for E to obtain a semiclassical approximation of the spectrum. This approximation should be valid asymptotically, that is, as $E \to \infty$, as we expect $I_0(E) \gg I_2(E) \gg I_4(E)\cdots$. We can check that this is indeed the case for $\alpha < 1$ as $I_0(E) \propto E$, $I_2(E) \propto E^{-5/2}$ and $I_4(E) \propto E^{-7/2}$. Note that for the particular value $\alpha = 0$ (or $\beta = 0$), the potential (24) becomes a harmonic potential and the corrections I_2 and I_4 vanish. It can be shown that all higher corrections I_{2n} vanish as well [18], which establishes the well-known fact that, in this case, the EBK quantization leads to an exact result.

However, for $\alpha = 1$, the potential (24) becomes a singular isotonic potential (see section 2.3.1). The second and fourth corrections become energy independent in this limit. It can be shown by calculating the entire WKB series that all corrections are energy independent and that their summation leads to the exact quantization condition [20, 23]. We can check that the two corrective terms I_2 and I_4 we have obtained are the coefficients of the Taylor expansion for the non-integral Maslov index (see [25]) of the exact quantization condition in $\beta = 0$. Indeed, the exact quantization condition for the radial-harmonic oscillator (26) (or isotonic potential) reads (see for example [26])

$$E_n = \left(n + \frac{\mu}{4}\right)\omega,$$

where the non-integral Maslov index is given by

$$\mu = 3 + \frac{\omega}{\beta^2}\left(\sqrt{1 + \frac{\beta^4}{\omega^2}} - \frac{\beta^2}{\omega} - 1\right). \tag{65}$$

Expanding this last expression around $\beta = 0$ yields

$$E_n \sim \left(n + \frac{1}{2} + \frac{1}{8}\frac{\beta^2}{\omega} - \frac{1}{32}\frac{\beta^6}{\omega^3} + \cdots\right)\omega,$$

which is precisely what we obtain from (63) given that $\alpha = 1$ implies $Q = 1$ and $P = 8$.

This remark brings us to consider that WKB series should be more and more accurate as the scaling parameter β tends to zero, i.e. in the limit where the potential tends to the harmonic oscillator. This statement is confirmed by the numerical computations performed in the next sections.

4.2. *Family II*

We restrict our study to the nonsingular potentials (34) and (35) for which the WKB method leads to an asymptotic series as $E \to \infty$.[3] In what follows, we again evaluate I_2 and I_4 for

[3] For the family of singular potentials (33), the corrections $I_2(E)$ and $I_4(E)$ can be calculated analytically. For example, setting $\hbar = 1, \beta = 1$ and $\omega = \sqrt{2}$ in (33), we obtain $I_2(E) = -\frac{\sqrt{2}}{48}[\mathrm{I}_0(2\sqrt{E}) + \frac{\mathrm{I}_1(2\sqrt{E})}{2\sqrt{E}}]$, where $\mathrm{I}_n(z)$ is the modified Bessel function of order n as defined in [38]. A similar expression involving the I_n's up to order 3 can be obtained for $I_4(E)$. As $\to \infty$, $\mathrm{I}_n(z) \sim \mathrm{e}^z/\sqrt{2\pi z}$ (see formula 9.7.1 in [38]). Therefore, $I_2(E)$ and $I_4(E)$ grow exponentially fast as $E \to \infty$. We can see that they alternate in sign though. It is not clear to us whether the entire WKB series could be summed and would eventually be finite as $E \to \infty$. Numerical results for (33) seem to suggest that $E_n \sim (n + \frac{\mu(\beta)}{4})\hbar\omega$, when $n \to \infty$, with a Maslov index, $\mu(\beta)$, ranging from 2 (small β) to 3 (large β).

Table 1. Table of the coefficients $M_{m,n}(\xi)$ appearing in (67) for $\omega = \sqrt{2}$.

$M_{m,n}(\xi)$	$\xi = 2$	$\xi = 3$
$M_{2,1}$	1.9020×10^{-2}	6.3076×10^{-3}
$M_{2,2}$	1.7287×10^{-1}	5.5608×10^{-2}
$M_{4,1}$	7.0128×10^{-3}	1.9414×10^{-3}

$\omega = \sqrt{2}$. The S function of (34) and (35) reads

$$S_\xi(X) = \frac{1}{\xi}\tanh(X), \qquad \xi \in \{2, 3\}. \tag{66}$$

We recover the general expression for the potentials by means of the scalings (14) and (18). The frequency is brought back to the general value ω by choosing $\gamma = \omega/\sqrt{2}$.

4.2.1. Expressions for I_2 and I_4. Although $S_\xi(X)$ in (66) has a very simple form, we are not able to provide an analytical expression for $I_2(E)$ and $I_4(E)$. Then, we resort to their numerical evaluation as a function of E and use the corresponding functions to calculate the semiclassical spectrum.

The lack of exact analytical forms for $I_2(E)$ and $I_4(E)$ does not prevent us from extracting some useful information regarding their asymptotic behaviour as $E \to \infty$. According to the fact that WKB should provide the best results in this limit, we expect this analysis to yield the right correction to the harmonic levels at high energy. Using (52), (55) and a result regarding the asymptotic expansion of fractional integrals[4] obtained in [28], we easily derive the large-energy behaviour of $I_2(E)$ and $I_4(E)$ to be

$$\begin{aligned} I_2(\xi; E) &\sim \frac{M_{2,1}(\xi)}{E^{5/2}} + \frac{M_{2,2}(\xi)}{E^{7/2}} + o\left(\frac{1}{E^{7/2}}\right), \\ I_4(\xi; E) &\sim \frac{M_{4,1}(\xi)}{E^{7/2}} + o\left(\frac{1}{E^{7/2}}\right), \end{aligned} \tag{67}$$

where $\xi \in \{2, 3\}$ and $M_{m,n}(\xi)$ are some nonzero coefficients related to the Mellin transform of the functions appearing in the numerator of the fractional integrals involved in (52) and (55) (see [27] or [28] for more details). The numerical values of these coefficients for a frequency $\omega = \sqrt{2}$ are listed in table 1.

4.2.2. Asymptotic quantization condition. Our asymptotic expansion in E is truncated at order $E^{-7/2}$ in (67) because the term $I_6(E)$ would be needed to evaluate the proper coefficient

[4] The result contained in [28] is actually not correct. For $f \sim e^{-\alpha t}\sum_{m=0}^{\infty} d_m t^{-r_m}$, $t \to \infty$ and $\alpha > 0$, the generalized fractional integral

$$I^{\mu}_{\lambda^p} f(\lambda) = \frac{1}{\Gamma(\mu)}\int_0^{\lambda} (\lambda^p - \xi^p)^{\mu-1} p\xi^{p-1} f(\xi)\,d\xi$$

has an asymptotic behaviour given by

$$I^{\mu}_{\lambda^p} f(\lambda) \sim \sum_{n=0}^{\infty} \frac{pM[f; p(n+1)]}{n!\Gamma(\mu-n)}(-1)^n \lambda^{-p(n-\mu+1)}$$

as $\lambda \to \infty$. Here, $M[f; x] = \int_0^\infty f(u)u^{x-1}\,du$ is the Mellin transform of f. In [28], the first factor p appearing in the series above is missing due to an error in the calculation of some residue. The reader is referred to this paper for more details. This correct asymptotic expansion has been used to calculate the coefficients of table 1.

of order $E^{-9/2}$ for the entire WKB series. We note again in this example that WKB series are generally no *direct* asymptotic expansions in energy. Obtaining such a series requires $I_{2n}(E)$ to be expanded as asymptotic series before terms of equal power in E can be collected. A method has been developed in [29] which allows for the derivation of a proper (direct) asymptotic expansion in E. Its range of application seems to be restricted to power-law potentials, however, and it is not clear how it could be used in the present case (see [30, 31]).

Finally, applying the scaling (59) to (67), we obtain an asymptotic quantization condition for potentials (34) and (35) valid for any frequency ω and any parameter β such that $(\beta/\omega)^2 E$ is large

$$E + \frac{\omega^5}{4\beta^3}\frac{M_{2,1}}{E^{5/2}} + \frac{\omega^5}{4\beta}\left\{\frac{\omega^2}{2\beta^4}M_{2,2} + M_{4,1}\right\}\frac{1}{E^{7/2}} \sim \left(n+\frac{1}{2}\right)\omega, \qquad \frac{\beta^2 E}{\omega^2} \to \infty. \tag{68}$$

For simplicity, we have dropped the ξ dependence of the coefficients $M_{m,n}(\xi)$. The last condition in (68) is absolutely essential. It comes from the fact that, when deriving the asymptotic forms (67), we have assumed that E was large in $I_2(E)$ and $I_4(E)$. As (59) rescales the energy by a factor $(\beta/\omega)^2$, we have to consider that $(\beta/\omega)^2 E$ is now large. For example, this condition ensures that the corrective terms of order $E^{-5/2}$ and $E^{-7/2}$ tend to zero in (68) as $\beta \to 0$ (harmonic limit) providing $E > \beta^{-2}$.

5. Numerical evaluation of the spectrum

5.1. Numerical solution of singular Sturm–Liouville problems

As the difference between the spectrum of a general isochronous potential and the harmonic one is expected to be small, its numerical determination requires an accurate method.

Our numerical evaluation of the spectrum is based on the Sturm–Liouville problem (SLP) (58). We have solved this equation by means of a shooting method provided by the code SLEIGN2 developed by Bailey, Everitt and Zettl [33, 34]. Among other advantages, solving (58) only requires the analytical expression for $S(X)$ and not the potential. It can then be easily used for a wide range of potentials.

The SLP (58) is always singular given that its endpoints are located at $X = \pm\infty$ (where the function $\phi(X)$ vanishes). The type of singularity may vary according to the potential under consideration. But in any case, it is the same for the SLP (58) and the associated Schrödinger equation (56) (see lemmas 1 and 2 of [36]).

5.2. Endpoint classification

Running SLEIGN2 requires knowledge of the singularity type of each endpoint of the SLP under investigation. We then classify hereafter those susceptible to be met in handling isochronous potentials. To determine the type of singularity of each branch of $V(x)$, we first note that

Claim 6. *If $V(x)$ is isochronous of frequency ω*

$$\forall x \in \mathbb{R}, \qquad V(x) \geqslant \frac{\omega^2}{8}x^2. \tag{69}$$

This is a consequence of the fact that $|S(X)| < 1 \Rightarrow |\bar{x}(X)| \leqslant |X|$. Now, $x = X + \bar{x}(X) \Rightarrow |x| \leqslant 2|X|$, hence $\omega^2x^2/8 \leqslant \omega^2X^2/2 = V(x)$.

Claim 7. *An endpoint x_e of $V(x)$ located at $\pm\infty$ is always of both the limit-point and non-oscillatory types.*

Indeed, applying theorem 7 of [36] and claim 6 proves this endpoint to be of the *limit-point* type. Moreover, as $\lim_{x\to x_e} V(x) = \infty$, applying theorem 6 of the same paper shows that x_e is of the *non-oscillatory* type.

Claim 8. *Assume an endpoint x_e of $V(x)$ is located at x_0, where x_0 is finite. Then, it is always of the non-oscillatory type.*

- *If* $\lim_{x\to x_0} \inf(x - x_0)^2 V(x) > 3/8$ *or if* $(x - x_0)^2 V(x) \geqslant 3/8$ *and* $\lim_{x\to x_0} \inf(x - x_0)^2 V(x) = 3/8$*, then the singularity is of the limit-point type.*
- *If* $\lim_{x\to x_0} \sup(x - x_0)^2 V(x) < 3/8$*, it is of the limit-circle type.*

Indeed, as $\lim_{x\to x_0} V(x) = +\infty$, applying theorem 4 of [36] shows immediately that the type of singularity of the branch located at x_0 is always *non-oscillatory*. Whether it is of the *limit-point* or *limit-circle* type relies on theorem 5 of the same paper.

5.3. Family I and family II potentials

Now, we investigate the spectrum of real-analytic potentials belonging to family I or II numerically and compare it to the WKB predictions. We select two specific types of potentials for this study with a frequency set to unity.

For potentials of type I, we consider (24) with $\alpha = 1/2$ ($\alpha < 1$ implies that the potential is nonsingular) and we denote it

$$V_{\mathrm{I}}(\beta; x) = \frac{2}{\beta^2}\left((\beta x + 1) - \sqrt{\frac{\beta x}{2}(\beta x + 2) + 1}\right)^2. \tag{70}$$

Of course, some quantitative differences are to be expected according to the value of α but, qualitatively, the features we are going to discuss hereafter remain unchanged.

For potentials of type II, we select (35) and denote it

$$V_{\mathrm{II}}(\beta; x) = \frac{1}{8\beta^2}\left[\ln\left(\frac{\sqrt{1 + 8\,\mathrm{e}^{3\beta x}} - 1}{2}\right)\right]^2. \tag{71}$$

Again, we could have chosen (34) instead without consequences regarding the qualitative behaviour of the spectrum as a function of the parameter β.

5.3.1. Small β regime. As already noted in the specific example of the isotonic potential (see section 4.1.2), the WKB method is expected to provide the best results in the limit where the parameter β is small. In this limit, the potential is close to being harmonic. We verify this statement by comparing the exact numerical corrections to the harmonic levels to those obtained from the semiclassical quantization condition. For $V_{\mathrm{I}}(\beta; x)$, we use the analytical expression (63), whereas for $V_{\mathrm{II}}(\beta; x)$ we resort to a numerical evaluation of $I_2(E)$ and $I_4(E)$. Within this section, $\beta = 1/2$.

We define the exact (numerical) correction to the harmonic levels by

$$\varepsilon_n = E_n - \left(n + \tfrac{1}{2}\right) \tag{72}$$

and, according to (45) and (59), its (continuous) fourth-order semiclassical approximation by

$$\varepsilon_{\mathrm{WKB}}(E) = -\beta^2 I_2(\beta^2 E) - \beta^6 I_4(\beta^2 E). \tag{73}$$

In this last expression, $I_2(E)$ and $I_4(E)$ are given by (52) and (55), respectively, and are evaluated at $\omega = 1$ for the functions $S(X)$ defined in (22) (type I) and (29) (type II). For $V_{\mathrm{I}}(\beta; x)$, (63) yields

$$\varepsilon_{\mathrm{WKB}}(E) = \frac{1}{2^5}\frac{\beta^2}{Q^{5/2}} - \frac{\beta^6}{2^{12}}\frac{P}{Q^{15/2}}, \tag{74}$$

where P and Q, defined in (64), are taken at $\alpha = 1/2$ and $\omega = 1$.

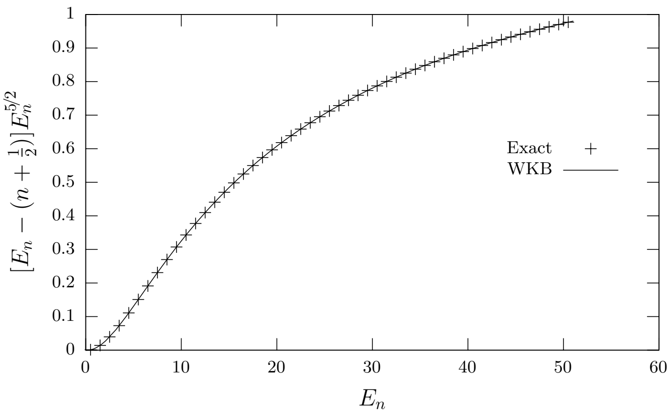

Figure 1. Comparison of the exact and semiclassical corrections to the harmonic spectrum for the first 50 energy levels of $V_{\mathrm{I}}(1/2; x)$ (see (70). Energy differences ε_n and $\varepsilon_{\mathrm{WKB}}(E)$ (see (72) and (74) are multiplied by $E^{5/2}$. Cross symbols are the numerical results and the solid line indicates the WKB results.

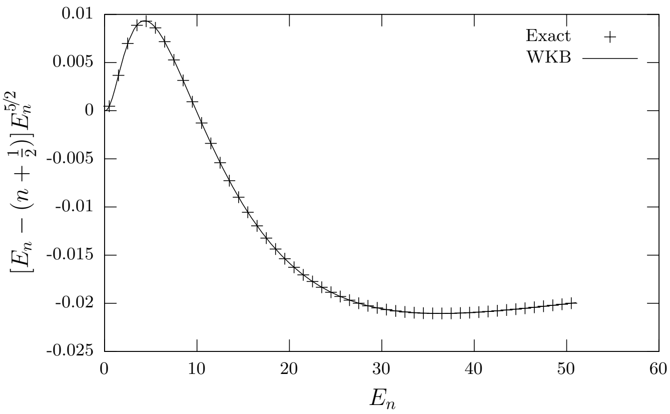

Figure 2. Comparison of the exact and semiclassical corrections to the harmonic spectrum for the first 50 energy levels of $V_{\mathrm{II}}(1/2; x)$ (see (71) multiplied by $E^{5/2}$ as in figure 1. Cross symbols are the numerical results and the solid line represents the result of the numerical evaluation of (73).

The results are presented in figures 1 and 2. For the sake of visibility, in both cases, the correction ε is multiplied by the inverse of its asymptotic decay, $E^{5/2}$ (see (61) and (68). Cross symbols represent the exact numerical results and the solid line indicates the fourth-order WKB results. As we can see, the fit is virtually perfect for the two spectra. In the limit where β is small, WKB provides corrections which are very accurate even in the lowest part of the spectrum. And this is so even though, in both cases, $E^{5/2}\varepsilon_{\mathrm{WKB}}(E)$ is far from reaching its asymptotic values given by $\sqrt{2}$ (type I) and $-2M_{2,1} \simeq -1.261$ (type II), respectively. In these precise examples, we see that WKB carry much more information than a simple high-energy asymptotic expansion.

5.3.2. Large β regime. As shown in claim 3, in the limit of large β, an isochronous potential asymptotically converges towards a split-harmonic oscillator. Thus, we expect the corresponding quantum spectra to be close. For this reason, an analysis of the spectrum of

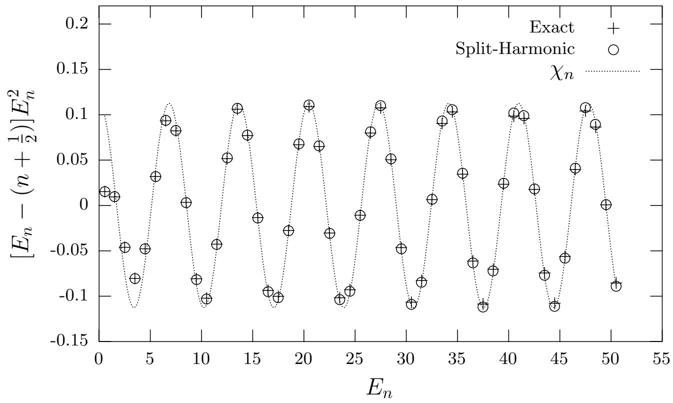

Figure 3. Comparison of the exact and split-harmonic corrections to the harmonic spectrum for the first 50 energy levels of $V_{\rm I}(50; x)$ (see (70)). Note that, contrary to figure 1, corrections are now multiplied by E^2. Cross symbols are the numerical results, circles are the corresponding split-harmonic corrections and the dotted line is the analytical approximation χ_n obtained in (B.3) (last expression).

the split-harmonic potential has been done in appendix B where a high-energy asymptotic expression for its quantum levels is derived. Using these results, we now compare the spectrum of potentials $V_{\rm I}(50; x)$ and $V_{\rm II}(30; x)$ to the spectrum of their respective asymptotic split-harmonic potentials ($\beta \to \infty$).

According to (27), when $\beta \to \infty$, $V_{\rm I}(\beta; x)$ converges to a split-harmonic oscillator with left and right frequencies whose ratio is given by $\rho_{\rm I} = (1 - \sqrt{\alpha})/(1 + \sqrt{\alpha}) = 3 - 2\sqrt{2}$. Regarding $V_{\rm II}(\beta; x)$, this ratio becomes $\rho_{\rm II} = 1/2$ (see end of section 4.2).

Again, for the sake of clarity, we have multiplied (72) by the inverse of its *split-harmonic* asymptotic decay. Note that the latter is different from the WKB decay. Indeed, as shown in (B.2), the *split-harmonic correction* to the harmonic levels vanishes like E^{-2} as E increases whereas the *WKB correction* decays like $E^{-5/2}$ instead.

In figure 3, even though ε_n behaves erratically, in contrast to the regularity observed in figure 1, the exact numerical results for $V_{\rm I}(50; x)$ are very well approximated by the split-harmonic corrections for low-energy levels. The apparent complex behaviour of ε_n is easily understood from expressions (B.2) and (B.3) of appendix B which show that, once multiplied by E_n^2, the correction (χ_n) is a periodic function of E_n (or $n + 1/2$) sampled at an incommensurate frequency. The dotted line of figure 3, which is a continuous version of χ_n (last expression of (B.3)), shows how accurate this asymptotic result is.

In figure 4, we have reported the results obtained for $V_{\rm II}(30; x)$. For such a large value of β, the correction to the harmonic levels is again very close to the split-harmonic correction. But this time, due to the rational value of $\rho_{\rm II} = 1/2$, (B.3) shows that $E_n^2 \varepsilon_n \sim \chi_n$ takes on two values only, $\frac{3^3}{2^{10}\pi}$ and $-\frac{3^3}{2^9\pi}$, in perfect agreement with those observed in figure 4.

Yet, both figures 3 and 4 indicate that the exact correction ε_n and its asymptotic approximation progressively split up as the energy increases. This is what is expected if the WKB analysis becomes exact as $E \to \infty$. Then, a crossover should exist between the two different power-law decays: E^{-2}, which is valid at low energy and $E^{-5/2}$, which is valid at high energy. This is difficult to observe for large values of β, however, because it would require a very high numerical accuracy when going up the spectrum. Instead, we try to observe this transition for an intermediate value of the parameter β in the next section.

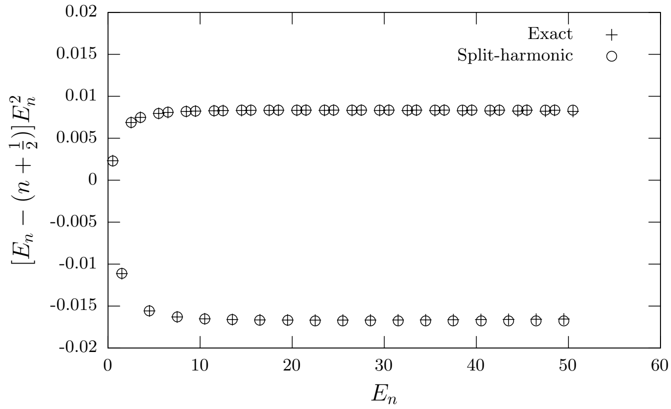

Figure 4. Comparison of the exact and split-harmonic corrections to the harmonic spectrum for the first 50 energy levels of $V_{\rm II}(30; x)$ (see (71)). As in figure 3, corrections are multiplied by E^2. Cross symbols are the numerical results and circles are the corresponding split-harmonic corrections.

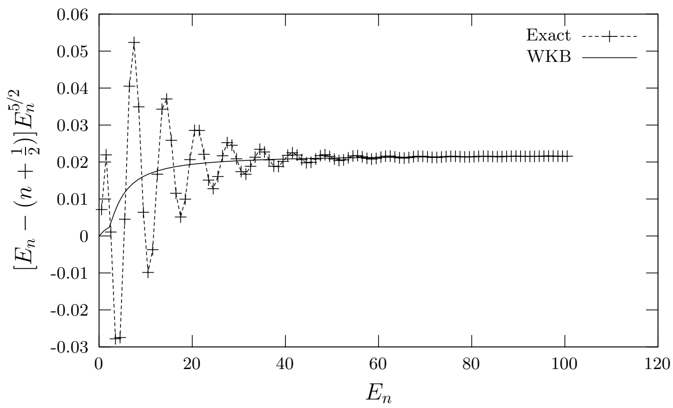

Figure 5. Comparison of the exact and semiclassical corrections to the harmonic spectrum for the first 100 energy levels of $V_{\rm I}(2; x)$ (see (70)). Energy differences ε_n and $\varepsilon_{\rm WKB}(E)$ (see (72) and (74)) are multiplied by $E^{5/2}$. Cross symbols linked by a dashed line are the numerical results and the solid line indicates the WKB results.

5.3.3. Intermediate β regime. In this last section, we select the two potentials $V_{\rm I}(2; x)$ and $V_{\rm II}(\sqrt[3]{2}; x)$. The results are shown in figures 5 and 6, respectively. The correction ε_n is again multiplied by the inverse of its asymptotic WKB decay, $E_n^{5/2}$. In both cases, we observe some oscillations at low energy which are clearly reminiscent of those produced by a split-harmonic oscillator. It can be checked that they are almost of the same frequency as those observed in figures 3 and 4. But at higher energy, they are damped out and converge towards the asymptotic behaviour predicted by the semiclassical approach. Indeed, from (74) and (68), we easily obtain $E^{5/2}\varepsilon_{\rm WKB} \to \sqrt{2}/64$ for $V_{\rm I}(2; x)$ and $E^{5/2}\varepsilon_{\rm WKB} \to -M_{2,1}/8$ for $V_{\rm II}(\sqrt[3]{2}; x)$. Hence, the asymptotic values 2.21×10^{-2} and -7.88×10^{-4} observed in figures 5 and 6, respectively.

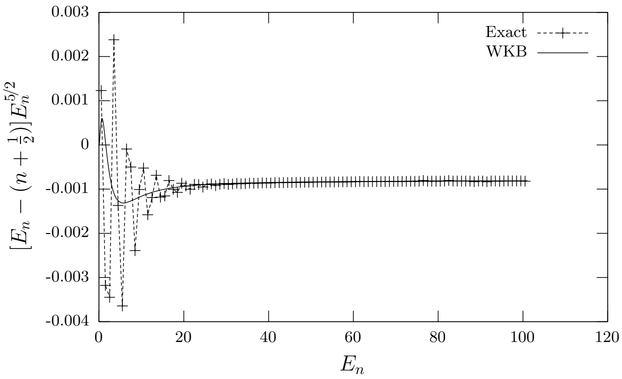

Figure 6. Comparison of the exact and semiclassical corrections to the harmonic spectrum for the first 100 energy levels of $V_{\mathrm{II}}(\sqrt[3]{2}; x)$ (see (71)) multiplied by $E^{5/2}$ as in figure 5. Cross symbols linked by a dashed line are the numerical results and the solid line represents the result of the numerical evaluation of (73).

6. Discussion and conclusion

In this paper, we have provided a quantitative way to analyse the spectrum of analytic isochronous potentials. As is already known, the spectrum of an isochronous potential is generally not strictly regularly spaced, in contrast to the harmonic one. Therefore, a method is required to obtain the possible corrections to the harmonic equispacing of its energy levels.

We have shown that a WKB analysis going beyond the semiclassical EBK quantization is precise enough to successfully account for these small differences. It provides the right asymptotic behaviour of these corrections at high energy and, in certain cases, is even able to describe them accurately from the ground state.

Moreover, we have seen that, for any isochronous potential $V(x)$, there exists a simple scaling transformation, namely $\tilde{V}(\beta; x) = V(\beta x)/\beta^2$, which preserves its isochronism as well as its frequency. Under certain regularity conditions, the one-parameter family $\tilde{V}(\beta; x)$ interpolates continuously between the harmonic oscillator ($\beta \to 0$) and a split-harmonic oscillator ($\beta \to \infty$). General spectral features, with respect to the parameter β, have been sketched in several examples.

When β is small, that is, when $\tilde{V}(\beta; x)$ is close to a harmonic oscillator, a higher order quantization condition derived from the WKB series accurately describes the entire spectrum, even its lowest part. When β is large, $\tilde{V}(\beta; x)$ is close to a split-harmonic oscillator and the lowest part of its spectrum essentially reproduces the split-harmonic features. Nevertheless, at high energy, the spectrum becomes asymptotically described by a higher order WKB quantization condition again, which significantly differs from the exact split-harmonic spectrum.

Corrections to the harmonic spectrum, that is, $\varepsilon_n = E_n - \left(n + \frac{1}{2}\right)\hbar\omega$, have been shown to scale like E_n^{-2} at high energy for a split-harmonic oscillator. Those generated by a higher order WKB method, however, scale generically like $E_n^{-5/2}$. Assuming that the latter ultimately represents the right high-energy asymptotic behaviour of the spectrum, this leads to the conclusion that a crossover between these two trends exists. Although its precise location within the spectrum is difficult to explore numerically, the spectrum of $\tilde{V}(\beta; x)$ for intermediate values of the parameter β supports evidence of such a transition.

Note finally that, in the examples treated in the last section, the first term of the WKB series going beyond the EBK quantization, namely I_2, dictates the leading order of the asymptotic decay of the correction ε_n as $E_n \to \infty$. Moreover, in these examples, $\varepsilon_n \sim E_n^{-5/2}$. We would like to point out that this need not always be the case. Indeed, starting from expression (52) for $I_2(E)$ and using the properties of Abel transforms [27, 37], it is possible to invert the problem and to calculate the general expression of the function $S(X)$ corresponding to a prescribed function $I_2(E)$ (see appendix C).

Using this result, we can choose $I_2(E)$ (and deduce the corresponding analytic isochronous potential via the function $S(X)$) such that its asymptotic decay is faster than the asymptotic decay of $I_4(E)$. An example given in appendix C proves that it is possible to find analytic potentials such that $I_2(E) \sim E^{-9/2}$ and $I_4(E) \sim E^{-7/2}$ as $E \to \infty$. The asymptotic leading order is then given by I_4 in this case and is different from the generic $E^{-5/2}$ decay. This precise example indicates how difficult it is to draw some general conclusions regarding the behaviour of the WKB series with respect to the energy.

Our last remark concerns the class of isochronous potentials with a strictly equispaced (harmonic) spectrum. We have met such a one-parameter family in section 4.1.2 called *isotonic* or *radial-harmonic* oscillator. Among other interesting properties, it has been proved that the isotonic potential is the only *rational* isochronous potential [39]. An interesting question is whether this family is the only one to be both classical and quantum 'harmonic'. Without this being a proof, we would like to mention the following interesting result:

Claim 9. *The most general family of analytic isochronous potentials, such that all the terms of the WKB series, $I_{2n}(E), n \geqslant 1$, defined in (46), are constant (energy independent), is the family of the isotonic oscillator with potential*

$$V(\beta; x) = \frac{\omega^2}{8\beta^2}\left(\beta x + 1 - \frac{1}{\beta x + 1}\right)^2, \qquad x > -\frac{1}{\beta}.$$

Proof. As already proved in [20, 23], all the terms $I_{2n}(E), n \geqslant 1$ of the isotonic potential are constant and once summed, the WKB series leads to the exact quantization condition for this potential. To prove that this family is the most general, we use the result of appendix C, which shows that requiring $I_2(E)$ to be constant implies that $S(X)$ corresponds to the function of the isotonic oscillator. □

Obviously, the fact that the $I_{2n}, n \geqslant 1$, are constant ensures that the spectrum is strictly equispaced. Indeed, knowing that $I_0(E)$ is proportional to E, because the potential is isochronous and that all other terms (I_1 included) are constant, immediately leads to a quantization condition of the form $E = \alpha(n+\mu)$, $(\alpha, \mu) \in \mathbb{R}^2$ and, hence, to a regular spacing between consecutive energy levels.

Unfortunately, the claim above is no proof that the isotonic family is the only one for which all terms of the WKB series add up to an energy-independent expression. For example, we could think of an overall cancellation of the energy-dependent terms generated by the $I_{2n}(E)$'s. Moreover, although the last condition is quite a natural way to ensure that the spectrum is strictly equispaced, the function $\sum_{n=1}^{\infty} I_{2n}(E)$ could be *energy dependent* and the quantization condition $\sum_{n=0}^{\infty} I_{2n}(E) = (n + \frac{1}{2})\hbar$ still has an equispaced spectrum for solution. Although they seem quite unlikely to us, these possibilities cannot be ruled out and the question of the class of potentials both classical and quantum harmonic remains open.

Acknowledgments

I am grateful to S Flach for having stimulated my interest in isochronous potentials, to R S MacKay for fruitful exchanges on the subject and to A Kalinowski for her useful comments and suggestions regarding this manuscript.

Appendix A. Scaling of the WKB terms

Here, we show the transformation affecting each term of the WKB series as the potential $V(x)$ is rescaled to $\tilde{V}(x) = (\gamma/\beta)^2 V(\beta x)$. First, we consider the Schrödinger equation for $\tilde{V}(x)$,

$$-\frac{\hbar^2}{2}\frac{\mathrm{d}^2\tilde{\psi}(x)}{\mathrm{d}x^2} + \tilde{V}(x)\tilde{\psi}(x) = \tilde{E}\tilde{\psi}(x). \tag{A.1}$$

According to (45), its WKB quantization condition reads

$$\sum_{k=0}^{\infty} \tilde{I}_{2k}(\tilde{E}) = \left(n + \frac{1}{2}\right)\hbar, \qquad n \in \mathbb{N}. \tag{A.2}$$

Given equation (46), $\tilde{I}_{2k}(\tilde{E})$ is proportional to $\hbar^{2k}$ and we will explicitly write it

$$\tilde{I}_{2k}(\tilde{E}) = \hbar^{2k}\tilde{\mathcal{I}}_{2k}(\tilde{E}). \tag{A.3}$$

Let $y = \beta x$, $\psi(y) = \tilde{\psi}(x)$. Equation (A.1) is immediately transformed to

$$-\frac{\hbar^2}{2}\left(\frac{\beta^2}{\gamma}\right)^2 \frac{\mathrm{d}^2\psi(y)}{\mathrm{d}y^2} + V(y)\psi(y) = E\psi(y), \tag{A.4}$$

where

$$E = \left(\frac{\beta}{\gamma}\right)^2 \tilde{E}. \tag{A.5}$$

Equation (A.4) is a Schrödinger equation in the potential $V(x)$ with an effective Planck's constant $\hbar_{\mathrm{eff}} = \frac{\beta^2\hbar}{\gamma}$. Therefore, its quantization condition is

$$\sum_{k=0}^{\infty} \left(\frac{\beta^2\hbar}{\gamma}\right)^{2k} \mathcal{I}_{2k}(E) = \left(n + \frac{1}{2}\right)\frac{\beta^2\hbar}{\gamma}, \qquad n \in \mathbb{N}. \tag{A.6}$$

Given that (A.1) and (A.4) are one and the same equation their quantization conditions are the same. Dividing (A.6) by $\frac{\beta^2}{\gamma}$ and identifying the series term by term to (A.2), we obtain the desired scaling

$$\tilde{I}_{2k}(E) = \left(\frac{\beta^2}{\gamma}\right)^{2k-1} I_{2k}\left(\frac{\beta^2 E}{\gamma^2}\right). \tag{A.7}$$

Another way to proceed is to consider the explicit expression for $I_{2n}(E)$ given in [22]. According to equation (44) of this reference, for $m \geqslant 1$, $I_{2m}(E)$ can be written as[5]

$$I_{2m}(E) = -\frac{\sqrt{2}}{\pi}\sum_{L(\nu)=2m} \frac{2^{|\nu|}J_\nu(E)}{(2m-3+2|\nu|)!!}, \tag{A.8}$$

where

$$J_\nu(E) = \frac{\partial^{m-1+|\nu|}}{\partial E^{m-1+|\nu|}} \int_{x_-(E)}^{x_+(E)} \frac{U_\nu V^{(\nu)}(x)}{\sqrt{E - V(x)}}\,\mathrm{d}x \tag{A.9}$$

and where $\nu = (\nu_1, \nu_2, \ldots, \nu_{2m})$, $\nu_j \in \mathbb{Z}^+$, $L(\nu) = \sum_{j=1}^{2m} j\nu_j$ and $|\nu| = \sum_{j=1}^{2m}\nu_j$. Moreover,

$$V^{(\nu)}(x) = \prod_{j=1}^{2m}\left(\frac{\mathrm{d}^j V}{\mathrm{d}x^j}(x)\right)^{\nu_j}. \tag{A.10}$$

[5] Differences in the coefficients of this formula when compared to equation (44) of [22] are due to a different choice in the normalization of the initial Schrödinger equation.

Coefficients U_ν are defined by a recurrence relation but are not given here since they are not needed in deriving the result.

Let us evaluate $\tilde{I}_{2m}(E)$ which is related to the potential $\tilde{V}(x) = (\gamma/\beta)^2 V(\beta x)$. We first see that

$$\tilde{V}^{(\nu)}(x) = \left(\frac{\gamma}{\beta}\right)^{2|\nu|} \beta^{L(\nu)} V^{(\nu)}(\beta x). \tag{A.11}$$

Taking this result and (15) into account, we obtain

$$\tilde{J}_{(\nu)}(E) = \frac{1}{\gamma}\left(\frac{\beta}{\gamma}\right)^{2(m-1)} \beta^{L(\nu)} J_{(\nu)}\left(\frac{\beta^2 E}{\gamma^2}\right). \tag{A.12}$$

Given that in equation (A.8), the sum is restricted to terms such that $L(\nu) = 2m$, we finally re-obtain (A.7).

Appendix B. Spectrum of the split-harmonic potential

The *split-harmonic* potential is made of two half-parabolic arches connected in $x = 0$ with different frequencies ω_l and ω_r to the left and to the right, respectively. It is an isochronous potential with frequency $\omega = 2\omega_\mathrm{l}\omega_\mathrm{r}/(\omega_\mathrm{l} + \omega_\mathrm{r})$. It turns out that an exact expression for its quantization condition is available and reads (see for example [1])

$$\frac{\sqrt{\rho}}{\Gamma\left(\frac{3}{4} - \rho x\right)\Gamma\left(\frac{1}{4} - x\right)} + \frac{1}{\Gamma\left(\frac{1}{4} - \rho x\right)\Gamma\left(\frac{3}{4} - x\right)} = 0. \tag{B.1}$$

In this expression, $\rho = \omega_\mathrm{l}/\omega_\mathrm{r}$ is the ratio between the left and right frequencies and without loss of generality, we restrict its values to the range $[0, 1]$ (values within the range $[1, \infty]$ give the same spectrum because they amount to flipping the potential around its vertical axis). Finally, $x = (\nu + 1/2)/(1 + \rho)$ and $E_\nu = \left(\nu + \frac{1}{2}\right)\omega$. Thus, once (B.1) is solved for x, we know the energy levels E_ν.

There are two limiting cases corresponding to the *harmonic* potential ($\rho = 1$) and the *half-harmonic* potential ($\rho = 0$) that can be treated exactly. As $1/\Gamma(z)$ is an entire function in $\mathbb{C}$ which vanishes at all negative integers, we obtain

- For $\rho = 1$, $x = 3/4 - n$ or $x = 1/4 - n$, $n \in \mathbb{N}$. Thus, $\nu = n \in \mathbb{N}$ and $E_n = \left(n + \frac{1}{2}\right)\omega$ which is indeed the spectrum of the harmonic oscillator.
- For $\rho = 0$, $x = 3/4 - n$, $n \in \mathbb{N}$. Thus, $\nu = n + 1/4$ and $E_n = \left(n + \frac{3}{4}\right)\omega$ which is also known to be the spectrum of the harmonic oscillator on the half-line.

In these two limiting cases, energy levels are exactly equidistant within the spectrum. The only noticeable change regards the Maslov index of the half-harmonic oscillator which is 3/4 due to the presence of a 'wall-type' singularity in $x = 0$ (see for example [25]).

However, for $\rho > 0$, no such singularity occurs. We expect the spectrum to be asymptotically given by the semiclassical energy levels, that is $E_n \sim \left(n + \frac{1}{2}\right)\omega$ as $n \to \infty$. As it has already been shown [1], the spectrum of the split-harmonic potential is not regularly spaced for $0 < \rho < 1$. Approximate expressions for the energy levels are given in [1] when $\rho \sim 1$ or $\rho \sim 0$.[6] Explicit corrections are given for the first five energy levels when $\rho \sim 1$ and it is noted that '*the striking feature of these results is that they alternate in sign*'.

To explain this interesting phenomenon, we perform an asymptotic expansion of the energy levels E_n for $\rho > 0$ as $\to \infty$. Starting from (B.1) and after some algebra,

[6] These authors use the variable ξ related to ρ by $\rho = (1 - \xi)/(1 + \xi)$.

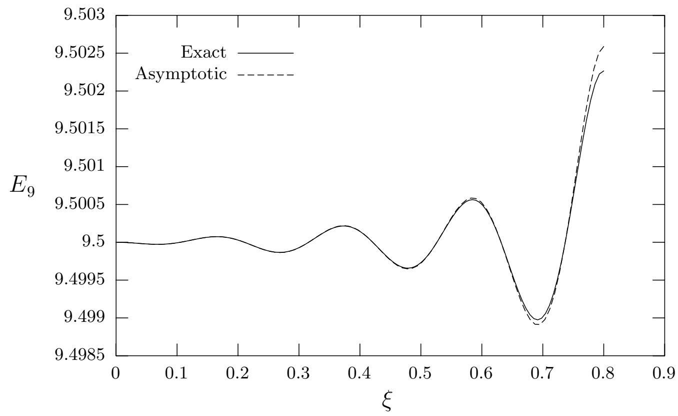

Figure 7. Parametric evolution of the ninth energy level, E_9, of a split-harmonic oscillator with frequency ratio ρ with respect to the variable $\xi = (1-\rho)/(1+\rho)$ $(\omega = 1)$. The solid line indicates the exact numerical result obtained from (B.1) and the dashed line indicates the asymptotic result (B.2) (to be compared with figure 3 of [1]).

we obtain

$$E_n \sim \left[n + \frac{1}{2} + \frac{\chi_n}{\left(n+\frac{1}{2}\right)^2} \right] \omega, \qquad \rho n \to \infty, \tag{B.2}$$

where

$$\begin{aligned} \chi_n &= -\frac{(1+\rho)^3(1-\rho)}{128\pi\rho^2} \cos\left[\frac{2\pi}{1+\rho}\left(n+\frac{1}{2}\right)\right] \\ &= \frac{(1+\rho)^3(1-\rho)}{128\pi\rho^2} \cos\left[\frac{2\pi\rho}{1+\rho}\left(n+\frac{1}{2}\right)\right]. \end{aligned} \tag{B.3}$$

Clearly, (B.2) indicates that E_n converges asymptotically to the harmonic levels for all $\rho > 0$. This could seem surprising since we know that they do not for $\rho = 0$ (they are equal to $(n+\frac{3}{4})\omega$ instead). In this respect, this limit is clearly singular. Hence, the requirement $\rho n \to \infty$ for (B.2).

The second remark is that, compared to *family I* and *family II* potentials whose levels converge like $E^{-5/2}$ towards the harmonic ones, the levels of the split-harmonic oscillator converge like E^{-2} instead. For isochronous potentials with large parameter β (that is, close to a split-harmonic oscillator), we expect a crossover between an asymptotic E^{-2} convergence law in the low/medium part of the spectrum and an $E^{-5/2}$ convergence law in the large energy limit.

Finally, equation (B.3) explains the oscillatory behaviour of the split-harmonic levels noted by Stillinger and Stillinger in [1] as both n and ρ vary since they are involved in the cosine function of (B.3). In particular, it can be checked that figure 3 of [1] is reproduced perfectly well by (B.3) (see figure 7). And the $n/2$ oscillations of $E_n(\rho)$ within the range $\rho \in [0, 1]$ are also explained by (B.3).

In figures 8 and 9, we give two typical corrections to the harmonic spectrum for $\rho = 0.9$ and $\rho = 0.3$, respectively. We define $\chi_n^{\text{exact}} = \left(n+\frac{1}{2}\right)^2\left[E_n^{\text{exact}} - \left(n+\frac{1}{2}\right)\right]$ and compare it with χ_n given by (B.3) $(\omega = 1)$. Exact values for the energy levels are obtained by solving (B.1) numerically. We see that χ_n^{exact} and χ_n are in good agreement even for small values of n. But as a general trend, this agreement degrades as $\rho \to 0$.

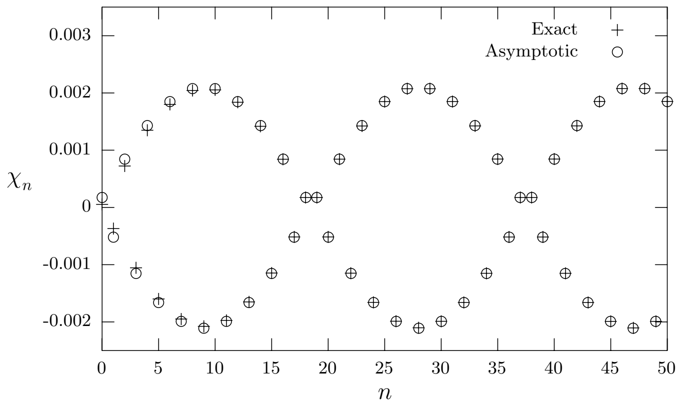

Figure 8. Corrections to the harmonic levels versus the energy level n, $\chi_n = (n+\frac{1}{2})^2[E_n-(n+\frac{1}{2})]$, for a split-harmonic potential with frequency ratio $\rho = 0.9$. Cross symbols represent the exact numerical result obtained by solving (B.1) and circles represent the asymptotic result χ_n given by (B.3).

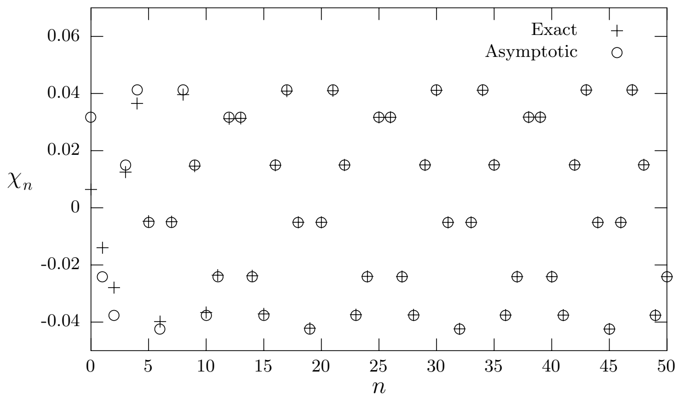

Figure 9. Same as figure 8 but for $\rho = 0.3$.

Appendix C. From $I_2(E)$ to $S(X)$

In this last appendix, we investigate the inverse problem which consists in recovering the potential (or at least the S function) corresponding to a prescribed function $I_2(E)$. Starting from expression (52) for $I_2(E)$, we have

$$\frac{1}{\sqrt{\pi}}\int_0^E \frac{g(v)}{(E-v)^{1/2}}\,\mathrm{d}v = -\frac{12\sqrt{\pi}E^2 I_2(E)}{\hbar^2\omega}, \tag{C.1}$$

where

$$g(v) = v^{3/2}\frac{\mathrm{d}^2}{\mathrm{d}v^2}\left[\frac{v}{1-S^2\left(\frac{\sqrt{2v}}{\omega}\right)}\right]. \tag{C.2}$$

Now, using the properties of Abel transforms (cf for example [27, 37]), we invert (C.1) to obtain

$$g(v) = -\frac{12}{\hbar^2\omega}\frac{\mathrm{d}}{\mathrm{d}v}\int_0^v \frac{E^2 I_2(E)}{(v-E)^{1/2}}\,\mathrm{d}E. \tag{C.3}$$

Provided the integral on the right-hand side of (C.3) can be evaluated, we can use (C.2) to calculate $S(X)$.

C.1. First example: $I_2(E) = \mathrm{cst}$

Let us assume that $I_2(E) = I_2 = \mathrm{cst}$. The right-hand side of (C.3) yields

$$g(v) = -\frac{32 I_2}{\hbar^2 \omega} v^{3/2}. \tag{C.4}$$

Reinstating this expression in (C.2), we find

$$\frac{v}{1 - S^2\left(\frac{\sqrt{2v}}{\omega}\right)} = -\frac{32 I_2}{\hbar^2 \omega}\left(\frac{v^2}{2} + av + b\right), \tag{C.5}$$

where a and b are two constants of integration. Since we are dealing with analytic potentials, S is continuous in 0 and as it is odd, $S(0) = 0$. Passing to the limit $v \to 0$ in (C.5) shows that $b = 0$. Hence,

$$1 - S^2\left(\frac{\sqrt{2v}}{\omega}\right) = -\frac{\hbar^2 \omega}{16 I_2}\frac{1}{v + 2a}. \tag{C.6}$$

Now, the condition $S(0) = 0$ determines the last constant, $a = -\hbar^2\omega/(32 I_2)$, and

$$S\left(\frac{\sqrt{2v}}{\omega}\right) = \frac{\sqrt{v}}{\sqrt{v - \frac{\hbar^2\omega}{16 I_2}}}, \tag{C.7}$$

which determines $S(X)$ for $X \geqslant 0$ and, as $S(X) = -S(-X)$, $S(X)$ on the entire real line. Finally, the requirement $|S(X)| < 1$ leads to the conclusion that I_2 has to be negative. Let us write $I_2 = -\hbar^2\beta^2/(8\omega)$ for some parameter $\beta \geqslant 0$, (C.7) simplifies to

$$S(X) = \frac{\beta X}{\sqrt{1 + \beta^2 X^2}}, \tag{C.8}$$

which is nothing but the S function of an isotonic potential with scaling parameter β (see (22), (26) and (18)).

C.2. Second example: $I_2(E) = -\frac{1}{6}\frac{\hbar^2\omega^8}{(\omega^2+2E)^{9/2}}$

Repeating what has been done in the previous example for the function $I_2(E)$ chosen above yields the following S function:

$$S(X) = \frac{2X[35 + 42X^2 + 15X^4]^{1/2}}{[105 + 455X^2 + 483X^4 + 165X^6]^{1/2}}. \tag{C.9}$$

This function is readily odd and $\forall X \in \mathbb{R}, |S(X)| < 1$. The corresponding potential has no singularity on the real line.

For (C.9), the asymptotic behaviour of the fourth WKB term $I_4(E)$, given by (55), is proportional to $E^{-7/2}$. To see this, we reinstate (C.9) into the function $G_1(X)$ defined in (54) and we perform an asymptotic analysis of the first term of (55). It turns out that $g_1(v) := v^{5/2}\frac{\mathrm{d}^3}{\mathrm{d}v^3}\{G_1(\sqrt{2v}/\omega)\} \sim v^{-3/2}$ as $\to \infty$ and the Mellin transform $M[g_1, 1] = \int_0^\infty g_1(v)\,\mathrm{d}v \neq 0$. Thus, according to formula (4.10.25) of [27], the first term of (55) scales like $E^{-7/2}$ as $E \to \infty$. It can be verified in the same way that the second term scales like $E^{-9/2}$ and is negligible when compared to the first one.

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